

STAT 545 Final Exam Notesheet

1 Stochastic Processes

Stochastic Process

A stochastic process is a collection of random variables

$$\{X_t : t \in I\}$$

where I is the index set (usually time) and S is the state space.

2 Markov Chains

Markov Property

$$P(X_{n+1} = j \mid X_0, \dots, X_n = i) = P(X_{n+1} = j \mid X_n = i)$$

2.1 Transition Matrix

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

$$P = \begin{pmatrix} p_{11} & p_{12} & \dots \\ p_{21} & p_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Properties:

- $p_{ij} \geq 0$
- $\sum_j p_{ij} = 1$

2.2 Time Homogeneous

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i)$$

3 n -Step Transition Probabilities

$$P_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$$
$$P^{(n)} = P^n$$

Chapman–Kolmogorov

$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)}$$

4 Distribution Evolution

If initial distribution is

$$\alpha = (\alpha_1, \dots, \alpha_k)$$

then

$$P(X_n = \cdot) = \alpha P^n$$

5 Stationary Distributions

$$\pi = \pi P$$

$$\pi_j = \sum_i \pi_i P_{ij}$$

with

$$\sum_i \pi_i = 1$$

6 Regular / Ergodic Chains

A matrix P is **regular** if

$$P^n > 0$$

for some n .

If P is regular:

- unique stationary distribution
- limiting distribution exists
- rows of P^n converge to π

Finite chain is **ergodic** if

- irreducible
- aperiodic

Then

$$P_{ij}^{(n)} \rightarrow \pi_j$$

7 Communication, Recurrence, Period

State j is accessible from i if

$$P_{ij}^{(n)} > 0$$

for some $n \geq 1$.

Communicating: accessible both ways.

Irreducible: all states communicate.

State j is recurrent if

$$P(\text{return to } j) = 1$$

Equivalent:

$$\sum_{n=0}^{\infty} P_{jj}^{(n)} = \infty$$

Transient if

$$\sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty$$

$$d(i) = \gcd\{n > 0 : P_{ii}^{(n)} > 0\}$$

Aperiodic if $d(i) = 1$.

8 Detailed Balance / Reversibility

$$\pi(i)P(i, j) = \pi(j)P(j, i)$$

Detailed balance $\Rightarrow \pi$ stationary.

Reversible chain $\iff$ detailed balance holds for some π .

9 Birth–Death Chains

$$P(i, i+1) = p_i, \quad P(i, i-1) = q_i$$

$$\frac{\pi(i+1)}{\pi(i)} = \frac{p_i}{q_{i+1}}$$

10 Reflecting Random Walks

Interior transitions:

$$P(i, i+1) = p, \quad P(i, i-1) = 1-p$$

Typical reflecting boundaries:

$$P(0, 1) = 1, \quad P(n, n-1) = 1$$

Let

$$r = \frac{p}{1-p}$$

Then from detailed balance,

$$\pi(i+1) = r \pi(i)$$

up to boundary adjustments / normalization.

For finite path with no self-loops: usually irreducible, period 2.

11 Mean Return Time

For positive recurrent state i ,

$$E[T_i] = \frac{1}{\pi(i)}$$

Long-run fraction of time in state i is $\pi(i)$.

12 Hitting Times / Hitting Probabilities

For set A ,

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}$$

For a single state j ,

$$\tau_j = \inf\{t \geq 0 : X_t = j\}$$

Hitting probability:

$$h_i = P_i(\tau_A < \infty)$$

satisfies

$$h_i = \sum_j P_{ij} h_j$$

for $i \notin A$, with boundary

$$h_i = 1, \quad i \in A$$

Expected hitting time:

$$m_i = E_i[\tau_A]$$

satisfies

$$m_i = 1 + \sum_j P_{ij} m_j$$

for $i \notin A$, with

$$m_i = 0, \quad i \in A$$

Try exponential form $h_i = \lambda^i$ for nearest-neighbor walks.

13 Recurrence Classification on $\mathbb{N}$

For reflecting walk on $\mathbb{N}$ with interior moves

$$P(i, i+1) = p, \quad P(i, i-1) = 1-p, \quad P(0, 1) = 1$$

$$r = \frac{p}{1-p}$$

$$p < \frac{1}{2} \Rightarrow \text{positive recurrent}$$

$$p = \frac{1}{2} \Rightarrow \text{null recurrent}$$

$$p > \frac{1}{2} \Rightarrow \text{transient}$$

Characteristic equation for return probs:

$$p\lambda^2 - \lambda + (1-p) = 0$$

14 Two-State Markov Chains

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad \alpha = (\alpha_0, \alpha_1)$$

14.1 Two-Step Transition

$$P^2 = \begin{pmatrix} (1-p)^2 + pq & p(2-p-q) \\ q(2-p-q) & (1-q)^2 + pq \end{pmatrix}$$

14.2 Expectation of the State

Since $X_n \in \{0, 1\}$,

$$E[X_n] = P(X_n = 1)$$

$$E[X_2] = (\alpha P^2)_1$$

14.3 Closed Form for P^n

$$P^n = \frac{1}{p+q} \begin{pmatrix} q + p(1-p-q)^n & p - p(1-p-q)^n \\ q - q(1-p-q)^n & p + q(1-p-q)^n \end{pmatrix}$$

$$\pi = \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$$

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix}$$

$$E[X_n] = \frac{p}{p+q} + \left(\alpha_1 - \frac{p}{p+q} \right) (1-p-q)^n$$

15 Random Walks on Graphs

A random walk on a graph moves uniformly to a neighboring vertex.

$$P_{ij} = \frac{1}{\deg(i)} \quad \text{if } i \sim j$$

Key steps:

1. Compute degree of each vertex.
2. Normalize each row by degree.

Stationary distribution for simple random walk on undirected graph:

$$\pi(i) = \frac{\deg(i)}{\sum_j \deg(j)}$$

15.1 Period in Graph Walks

$$d(i) = \gcd\{n \geq 1 : P_{ii}^{(n)} > 0\}$$

Facts:

- bipartite graph $\Rightarrow$ period 2
- self-loop often makes chain aperiodic

16 Absorbing Markov Chains

A state is absorbing if

$$P_{ii} = 1$$

Reorder states so

$$P = \begin{pmatrix} Q & R \\ 0 & S \end{pmatrix}$$

where Q : transient $\rightarrow$ transient, R : transient $\rightarrow$ closed/absorbing.

16.1 Fundamental Matrix

$$F = (I - Q)^{-1}$$

Interpretation:

F_{ij} = expected visits to transient state j starting from i

16.2 Expected Time to Absorption

$$E[T \mid X_0 = i] = \sum_j F_{ij}$$

16.3 Absorption Probabilities

$$B = FR$$

B_{ij} : probability of eventual absorption in closed/absorbing state j .

16.4 Limiting Matrix

If closed classes are aperiodic,

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} 0 & (I - Q)^{-1} R \Pi \\ 0 & \Pi \end{pmatrix}$$

Π : stationary distributions of closed classes.

17 Estimating Transition Matrices

Suppose transition counts N_{ij} are observed.

17.1 MLE

$$\hat{P}_{ij} = \frac{N_{ij}}{\sum_j N_{ij}}$$

(row-normalized counts)

17.2 Laplace Smoothing

$$\hat{p}_i = \frac{N_i + 1}{\sum_j (N_j + 1)}$$

Prevents zero probabilities.

17.3 Dirichlet Smoothing

Prior:

$$p \sim \text{Dir}(\alpha\mu)$$

Posterior mean:

$$\tilde{p}_i = \frac{N_i + \alpha\mu_i}{N + \alpha}$$

18 Poisson Process

For homogeneous Poisson process ($N(t)$):

- $N(0) = 0$
- independent increments
- stationary increments
- $N(t) \sim \text{Poisson}(\lambda t)$

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$E[N(t)] = \lambda t, \quad \text{Var}(N(t)) = \lambda t$$

18.1 MLE for Poisson Rate

$$\hat{\lambda} = \frac{N(T)}{T}$$

18.2 Exponential Waiting Times

$$T_1 \sim \text{Exp}(\lambda)$$

$$P(T_1 > t) = e^{-\lambda t}$$

$$E[T_1] = \frac{1}{\lambda}$$

If waiting for m additional arrivals,

$$E[\text{wait time}] = \frac{m}{\lambda}$$

18.3 Thinning

If each event is observed independently with probability q ,

$$\lambda_{\text{obs}} = q\lambda, \quad \lambda = \frac{\lambda_{\text{obs}}}{q}$$

19 Nonhomogeneous Poisson Process

Intensity $\lambda(t)$ varies with time.

$$E[N(b) - N(a)] = \int_a^b \lambda(t) dt$$

$$N(b) - N(a) \sim \text{Poisson}\left(\int_a^b \lambda(t) dt\right)$$

Probability of no arrivals on $[0, t]$:

$$P(N(t) = 0) = \exp\left(-\int_0^t \lambda(s) ds\right)$$

First arrival survival:

$$P(T_1 > t) = \exp\left(-\int_0^t \lambda(s) ds\right)$$

20 Brownian Motion

Standard Brownian motion (B_t):

- $B_0 = 0$
- independent increments
- $B_t - B_s \sim N(0, t - s)$ for $t > s$

$$E[B_t] = 0, \quad \text{Var}(B_t) = t$$

$$\text{Cov}(B_s, B_t) = \min(s, t)$$

21 Brownian Motion with Drift

$$X_t = \mu t + \sigma B_t$$

$$E[X_t] = \mu t, \quad \text{Var}(X_t) = \sigma^2 t$$

If observed at equal spacing Δt ,

$$\hat{\mu} = \frac{1}{n\Delta t} \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})$$

For 2D:

$$Z_t = \mu t + \Sigma^{1/2} B_t, \quad \hat{\mu} = \frac{Z_{t_n} - Z_{t_0}}{t_n - t_0}$$

22 Brownian Bridge

Standard Brownian bridge:

$$W_t = B_t - tB_1, \quad 0 \leq t \leq 1$$

Equivalent in distribution to $(B_t \mid B_1 = 0)$.

$$E[W_t] = 0$$

$$\text{Cov}(W_s, W_t) = \min(s, t) - st$$

For $X_t = \mu t + \sigma B_t$,

$$(X_t \mid X_1 = 0) \stackrel{d}{=} \sigma(B_t - tB_1)$$

with covariance

$$\sigma^2(\min(s, t) - st)$$

23 Conditional Normal

If (X, Y) is jointly normal, then

$$X \mid Y = y \sim N\left(\mu_x + \frac{\sigma_{xy}}{\sigma_y^2}(y - \mu_y), \sigma_x^2 - \frac{\sigma_{xy}^2}{\sigma_y^2}\right)$$

Useful with Brownian motion:

$$(B_t, B_1) \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t & t \\ t & 1 \end{pmatrix}\right)$$

24 Random Walk Drift in 2D

For steps E, W, N, S with probabilities p_E, p_W, p_N, p_S ,

$$E[\Delta Z] = (p_E - p_W, p_N - p_S)$$

If counts are N_E, N_W, N_N, N_S over n steps,

$$\hat{p}_E = \frac{N_E}{n}, \quad \hat{p}_W = \frac{N_W}{n}, \quad \hat{p}_N = \frac{N_N}{n}, \quad \hat{p}_S = \frac{N_S}{n}$$

$$\hat{\delta} = \left(\frac{N_E - N_W}{n}, \frac{N_N - N_S}{n}\right)$$

Sample mean increment:

$$\bar{\Delta} = \frac{1}{n} \sum_{k=1}^n (Z_k - Z_{k-1})$$

If each step lasts Δt ,

$$\hat{\mu} = \frac{\hat{\delta}}{\Delta t}$$

25 Bayesian Logistic Regression

Model:

$$P(Y_i = 1 \mid \beta) = \frac{e^{\beta x_i}}{1 + e^{\beta x_i}}, \quad \beta \sim N(0, \tau^2)$$

Posterior kernel:

$$p(\beta \mid y, x) \propto \left[\prod_{i=1}^n \frac{e^{y_i \beta x_i}}{1 + e^{\beta x_i}} \right] \exp\left(-\frac{\beta^2}{2\tau^2}\right)$$

Odds ratio for one-unit increase in x :

$$e^\beta$$

Approx. MAP = maximizer of log-posterior.

Exponentiate posterior quantiles of β to get credible interval for e^β .

26 Metropolis–Hastings

Target density $\pi(\theta)$, proposal $q(\theta^* \mid \theta)$.

Acceptance probability:

$$\alpha(\theta, \theta^*) = \min\left\{1, \frac{\pi(\theta^*) q(\theta \mid \theta^*)}{\pi(\theta) q(\theta^* \mid \theta)}\right\}$$

If proposal symmetric ($q(\theta^* \mid \theta) = q(\theta \mid \theta^*)$):

$$\alpha(\theta, \theta^*) = \min\left\{1, \frac{\pi(\theta^*)}{\pi(\theta)}\right\}$$

Random-walk proposal:

$$\theta^* \sim N(\theta, s^2)$$

Rule of thumb: moderate acceptance rate is usually good.

27 Gaussian Mixture Model

Latent variable form:

$$z_i \mid \pi \sim \text{Bernoulli}(\pi), \quad y_i \mid z_i, \mu_0, \mu_1, \sigma^2 \sim N(\mu_{z_i}, \sigma^2)$$

Priors:

$$\pi \sim \text{Beta}(a, b), \quad \mu_k \sim N(m_k, s_k^2), \quad \sigma^2 \sim \text{Inv-Gamma}(\alpha, \beta)$$

27.1 Full Conditional for z_i

$$P(z_i = 1 \mid \cdot) = \frac{\pi \phi(y_i; \mu_1, \sigma^2)}{(1 - \pi) \phi(y_i; \mu_0, \sigma^2) + \pi \phi(y_i; \mu_1, \sigma^2)}$$

27.2 Full Conditional for π

If $n_1 = \sum_i z_i$, $n_0 = n - n_1$,

$$\pi \mid z \sim \text{Beta}(a + n_1, b + n_0)$$

27.3 Full Conditional for μ_k

For $k \in \{0, 1\}$, let $n_k = |\{i : z_i = k\}|$, $\bar{y}_k$ sample mean in cluster k .

$$\begin{aligned}(s_k^*)^{-2} &= s_k^{-2} + \frac{n_k}{\sigma^2} \\ m_k^* &= (s_k^*)^2 \left(\frac{m_k}{s_k^2} + \frac{n_k \bar{y}_k}{\sigma^2} \right) \\ \mu_k \mid \cdot &\sim N(m_k^*, (s_k^*)^2)\end{aligned}$$

27.4 Full Conditional for σ^2

Let

$$SSE = \sum_{i=1}^n (y_i - \mu_{z_i})^2$$

$$\sigma^2 \mid \cdot \sim \text{Inv-Gamma} \left(\alpha + \frac{n}{2}, \beta + \frac{SSE}{2} \right)$$

27.5 One Gibbs Sweep

Update in order:

1. all z_i

2. π
3. μ_0
4. μ_1
5. σ^2

28 Fast Problem-Solving Checklist

Common moves on exams

- Stationary dist.: solve $\pi P = \pi$ or use detailed balance.
- Long-run fraction of time in state i : use $\pi(i)$.
- Mean return time: use $1/\pi(i)$.
- Hitting probs.: write recursion + boundary conditions.
- Expected hitting time: write $m_i = 1 + \sum_j P_{ij} m_j$.
- Absorbing chain: compute $F = (I - Q)^{-1}$, then FR .
- Poisson counts: integrate $\lambda(t)$.
- Brownian drift MLE: net displacement / total time.
- MH: posterior ratio $\times$ proposal ratio.
- Gibbs: identify full conditional families.