

STAT 545 HW 6

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Assignment 6: Homogeneous Poisson Processes

Exercise 6.12

Starting at noon, diners arrive at a restaurant according to a Poisson process at the rate of five customers per minute. The time each customer spends eating at the restaurant has an exponential distribution with mean 40 minutes, independent of other customers and independent of arrival times. Find the distribution, as well as the mean and variance, of the number of diners in the restaurant at 2 p.m.

Solution

Let B_t denote the number of diners in the restaurant at time t minutes after noon. Arrivals occur according to a Poisson process with rate $\lambda = 5$ per minute, and the time spent eating by each diner is i.i.d. exponential with mean 40 minutes. Thus the eating time Z has distribution $\text{Exp}(\mu)$ with $\mu = 1/40$ per minute.

We want the distribution of B_{120} .

Let N_t be the number of arrivals in $[0, t]$. Then $N_t \sim \text{Poisson}(\lambda t)$. Conditioning on N_t ,

$$\begin{aligned}\mathbb{P}(B_t = k) &= \sum_{n=k}^{\infty} \mathbb{P}(B_t = k \mid N_t = n) \mathbb{P}(N_t = n) \\ &= \sum_{n=k}^{\infty} \mathbb{P}(B_t = k \mid N_t = n) e^{-\lambda t} \frac{(\lambda t)^n}{n!}.\end{aligned}$$

Given $N_t = n$, let $U_1, \dots, U_n$ be the (unordered) arrival times in $(0, t)$. A standard property of the Poisson process states that conditional on $N_t = n$, the variables $U_1, \dots, U_n$ are i.i.d. $\text{Uniform}(0, t)$.

Let $Z_1, \dots, Z_n$ be the i.i.d. eating times, independent of $U_1, \dots, U_n$. Diner i is still in the restaurant at time t iff $U_i + Z_i > t$. Hence B_t equals the number of indices $i \in \{1, \dots, n\}$ such that $U_i + Z_i > t$.

Define the indicator $I_i = \mathbf{1}\{U_i + Z_i > t\}$. Then, given $N_t = n$,

$$B_t = \sum_{i=1}^n I_i.$$

Moreover, conditional on $N_t = n$, the pairs (U_i, Z_i) are i.i.d., so $I_1, \dots, I_n$ are i.i.d. Bernoulli with success probability

$$p = \mathbb{P}(U_1 + Z_1 > t).$$

Therefore,

$$\mathbb{P}(B_t = k \mid N_t = n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Since $U_1 \sim \text{Uniform}(0, t)$ and is independent of Z_1 ,

$$p = \mathbb{P}(U_1 + Z_1 > t) = \frac{1}{t} \int_0^t \mathbb{P}(Z_1 > t-x) dx.$$

Here $Z_1 \sim \text{Exp}(\mu)$, so $\mathbb{P}(Z_1 > y) = e^{-\mu y}$ for $y \geq 0$. Thus

$$p = \frac{1}{t} \int_0^t e^{-\mu(t-x)} dx = \frac{1}{t} \int_0^t e^{-\mu u} du = \frac{1 - e^{-\mu t}}{\mu t}.$$

Substituting $\mathbb{P}(B_t = k \mid N_t = n)$ into the conditioning sum gives

$$\begin{aligned}\mathbb{P}(B_t = k) &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t} \frac{p^k (\lambda t)^k}{k!} \sum_{n=k}^{\infty} \frac{((1-p)\lambda t)^{n-k}}{(n-k)!} \\ &= e^{-\lambda t} \frac{(p\lambda t)^k}{k!} \exp((1-p)\lambda t) \\ &= e^{-p\lambda t} \frac{(p\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots\end{aligned}$$

Hence

$$B_t \sim \text{Poisson}(p\lambda t).$$

Using $p = \frac{1 - e^{-\mu t}}{\mu t}$, we get

$$p\lambda t = \lambda \frac{1 - e^{-\mu t}}{\mu}.$$

With $\lambda = 5$, $\mu = 1/40$, and $t = 120$,

$$p\lambda t = \frac{5}{1/40} \left(1 - e^{-(1/40) \cdot 120}\right) = 200(1 - e^{-3}).$$

Therefore,

$$B_{120} \sim \text{Poisson}(200(1 - e^{-3})).$$

For a $\text{Poisson}(\theta)$ random variable, $\mathbb{E}[B_{120}] = \text{Var}(B_{120}) = \theta$, so

$$\mathbb{E}[B_{120}] = 200(1 - e^{-3}), \quad \text{Var}(B_{120}) = 200(1 - e^{-3}).$$

Exercise 6.23

Oak and maple trees are each located in the arboretum according to independent spatial Poisson processes with parameters λ_O and λ_M , respectively.

- In a region of x square meters, find the probability that both species of trees are present.
- In the arboretum, there is a circular pond of radius 100 m. Find the probability that within 20 m of the pond there are only oaks.

Spatial Poisson Process

A collection of random variables $(N_A)_{A \subseteq \mathbb{R}^d}$ is a spatial Poisson process with parameter λ if

- For each bounded set $A \subseteq \mathbb{R}^d$, N_A has a Poisson distribution with parameter $\lambda|A|$.
- Whenever A and B are disjoint sets, N_A and N_B are independent random variables.

Solution

Part A:

Let N_O be the number of oak trees and N_M be the number of maple trees in a region of area x (square meters). Since the locations follow independent spatial Poisson processes with intensities λ_O and λ_M ,

$$N_O \sim \text{Poisson}(\lambda_O x), \quad N_M \sim \text{Poisson}(\lambda_M x),$$

and N_O and N_M are independent.

The event that *both* species are present is

$$\{N_O \geq 1, N_M \geq 1\}.$$

Using complements and independence,

$$P(O_x \geq 1, M_x \geq 1) = P(O_x \geq 1) P(M_x \geq 1) = (1 - P(O_x = 0))(1 - P(M_x = 0)) = (1 - e^{-\lambda_O x})(1 - e^{-\lambda_M x}).$$

Part B:

Let A be the region consisting of all points within 20 m of the pond. Since the pond has radius 100 m, A is an annulus with inner radius 100 and outer radius 120, so

$$|A| = \pi(120^2 - 100^2).$$

Let O_A and M_A denote the numbers of oaks and maples in A . By the spatial Poisson assumptions,

$$O_A \sim \text{Poisson}(\lambda_O |A|), \quad M_A \sim \text{Poisson}(\lambda_M |A|),$$

and O_A and M_A are independent.

The event “within 20 m of the pond there are only oaks” means there are no maples in A , i.e. $\{M_A = 0\}$. Thus,

$$P(O_R \geq 1, M_R = 0) = P(O_R \geq 1) P(M_R = 0) = (1 - e^{-\lambda_O 4400\pi}) e^{-\lambda_M 4400\pi}.$$

Exercise 6.35

See the definitions for the spatial and nonhomogeneous Poisson processes. Define a nonhomogeneous, spatial Poisson process in $\mathbb{R}^2$. Consider such a process $(N_A)_{A \subseteq \mathbb{R}^2}$ with intensity function

$$\lambda(x, y) = e^{-(x^2+y^2)}, \quad -\infty < x, y < \infty.$$

Let C denote the unit circle, that is, the circle of radius 1 centered at the origin. Find $P(N_C = 0)$.

Nonhomogeneous Poisson Process

A counting process $(N_t)_{t \geq 0}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t)$, if

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a Poisson distribution with mean

$$\mathbb{E}(N_t) = \int_0^t \lambda(x) dx.$$

3. For $0 \leq q < r \leq s < t$, $N_r - N_q$ and $N_t - N_s$ are independent random variables.

Solution

For a nonhomogeneous spatial Poisson process with intensity function $\lambda(x, y)$, the number of points in a region A has a Poisson distribution with parameter

$$\iint_A \lambda(x, y) dx dy.$$

Hence,

$$P(N_A = 0) = \exp\left(-\iint_A \lambda(x, y) dx dy\right).$$

In our problem, C is the unit disk centered at the origin and

$$\lambda(x, y) = e^{-(x^2+y^2)}.$$

Therefore,

$$P(N_C = 0) = \exp\left(-\iint_C e^{-(x^2+y^2)} dx dy\right).$$

Since the region C is a disk and the intensity depends only on $x^2 + y^2$, we switch to polar coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta,$$

with Jacobian determinant r . On the unit disk,

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Thus,

$$\iint_C e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^1 e^{-r^2} r dr d\theta.$$

First integrate with respect to r . Let

$$u = r^2, \quad du = 2r dr,$$

so that $r dr = \frac{1}{2} du$. When $r = 0$, $u = 0$; when $r = 1$, $u = 1$. Hence,

$$\begin{aligned} \int_0^1 e^{-r^2} r dr &= \frac{1}{2} \int_0^1 e^{-u} du \\ &= \frac{1}{2} (1 - e^{-1}). \end{aligned}$$

Now integrate with respect to θ :

$$\begin{aligned} \int_0^{2\pi} \int_0^1 e^{-r^2} r dr d\theta &= \int_0^{2\pi} \frac{1}{2} (1 - e^{-1}) d\theta \\ &= 2\pi \cdot \frac{1}{2} (1 - e^{-1}) \\ &= \pi (1 - e^{-1}). \end{aligned}$$

Therefore,

$$P(N_C = 0) = \exp(-\pi(1 - e^{-1})).$$

Exercise 6.41

Goals occur in a soccer game according to a Poisson process. The average total number of goals scored for a 90-minute match is 2.68. Assume that two teams are evenly matched. Use simulation to estimate the probability both teams will score the same number of goals. Compare with the theoretical result.

Solution

Since the total number of goals in 90 minutes follows a Poisson process with mean 2.68, and the teams are evenly matched, each team's number of goals is an independent Poisson random variable with mean $2.68/2 = 1.34$.

Let X and Y denote the number of goals scored by the two teams. Then

$$X, Y \sim \text{Poisson}(1.34), \quad \text{independently.}$$

The probability that both teams score the same number of goals is

$$P(X = Y) = \sum_{k=0}^{\infty} P(X = k)P(Y = k) = \sum_{k=0}^{\infty} \left(\frac{e^{-1.34} 1.34^k}{k!} \right)^2.$$

Evaluating this sum gives

$$P(X = Y) \approx 0.259.$$

Thus, theoretically, the probability that both teams score the same number of goals is approximately 0.259.

Input:

Listing 1: Poisson Soccer Simulation

```
1 trials <- 100000
2 mean(rpois(trials, 1.34) == rpois(trials, 1.34))
```

Output: We get .25812

Exercise 6.42

Simulate the restaurant results of Exercise 6.12.

Solution

Input:

Listing 2: South Carolina Swamp Forest

```
1 trials <- 10000
2 sim <- numeric(trials)
3 for (i in 1:trials) {
4   cust <- rpois(1, 5*60*2) # no. of customers in 2 hours
5   arr <- sort(runif(cust, 0, 120))
6   times <- arr + rexp(cust, 1/40)
7   sim[i] <- sum(times > 120) # no of customers at 2 pm
8 }
9
10 mean(sim)
11
12 var(sim)
```

Output: Mean of 190.1929 and Variance of 192.4839

Exercise 6.43

Simulate a spatial Poisson process with $\lambda = 10$ on the box of volume 8 with vertices at the eight points $(\pm 1, \pm 1, \pm 1)$. Estimate the mean and variance of the number of points in the ball centered at the origin of radius 1. Compare with the exact values.

Solution

For a spatial Poisson process with intensity $\lambda = 10$ per unit volume, the number of points in any region A follows a Poisson distribution with mean

$$\lambda |A|.$$

The ball of radius 1 has volume

$$|A| = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$$

Hence the theoretical mean (and variance) of the number of points in the ball is

$$10 \cdot \frac{4}{3}\pi = \frac{40\pi}{3} \approx 41.89.$$

Thus, the exact distribution is

$$N \sim \text{Poisson}\left(\frac{40\pi}{3}\right),$$

so both the mean and variance equal $\frac{40\pi}{3} \approx 41.89$.

Input:

Listing 3: Exercise 6.43

```
1 trials <- 10000
2 sim <- numeric(trials)
3
4 for (k in 1:trials) {
5
6   npoints <- rpois(1, lambda = 10 * 8)
7
8   points <- matrix(runif(npoints * 3, -1, 1), ncol = 3)
9
10  sim[k] <- sum(rowSums(points^2) < 1)
11 }
12
13 mean(sim)
14 var(sim)
```

Output: Mean of 41.9851 and Standard Deviation of 42.03528.