

STAT 545 HW 3

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Assignment 3: Recurrence, Transience, and Periodicity

Exercise 3.23

Consider a k -state Markov chain with transition matrix

$$P = \begin{pmatrix} \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & \frac{1}{k} & \frac{1}{k} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Show that the chain is ergodic and find the limiting distribution.

Accessible

State j is said to be *accessible* from state i if $P_{ij}^n > 0$ for some $n \geq 0$. That is, there is positive probability of reaching j from i in a finite number of steps.

Communicate

States i and j *communicate* if i is accessible from j and j is accessible from i .

Communication Class

A communication class is a subset of the state space such that every pair of states in the subset communicate with each other, and no state in the subset communicates with any state outside the subset.

Irreducibility

A Markov chain is called *irreducible* if it has exactly one communication class. That is, all states communicate with each other.

Ergodic Markov Chain

A Markov chain is called *ergodic* if it is irreducible, aperiodic, and all states have finite expected return times. For finite Markov chains, the finiteness of expected return times always holds. Thus, a finite Markov chain is ergodic if and only if it is irreducible and aperiodic.

Ergodic Chains and Regular Matrices

Assume that P is the transition matrix of a finite Markov chain. The Markov chain is ergodic if and only if P is regular.

Fundamental Limit Theorem for Ergodic Markov Chains

Let $X_0, X_1, \dots$ be an ergodic Markov chain. There exists a unique, positive, stationary distribution π , which is the limiting distribution of the chain. That is,

$$\pi_j = \lim_{n \rightarrow \infty} P_{ij}^n \quad \text{for all } i, j.$$

Solution

Consider the k -state Markov chain with transition matrix P given above. Observe first that from state 1, the chain moves to any state $j \in \{1, \dots, k\}$ with probability $1/k > 0$. Thus, state 1 has positive probability of reaching every state in one step.

For states $i \geq 2$, the transition structure is deterministic:

$$P(i, i-1) = 1 \quad \text{for } i = 2, \dots, k,$$

so from any state i the chain reaches state 1 in exactly $i-1$ steps with probability 1.

Fix any state $i \in \{1, \dots, k\}$. In $i-1$ steps, the chain reaches state 1, and from state 1 it can move to any state j in one additional step with probability $1/k$. Hence, for all j ,

$$P^i(i, j) \geq \frac{1}{k} > 0.$$

It follows that the i -th row of P^i is strictly positive. Letting $n = \max\{1, 2, \dots, k\} = k$, we conclude that P^n has all entries strictly positive. Therefore, the chain is *regular*.

To find the limiting distribution, we solve for the stationary distribution $\pi = (\pi_1, \dots, \pi_k)$ satisfying

$$\pi = \pi P, \quad \sum_{j=1}^k \pi_j = 1.$$

From the structure of the transition matrix, we write the balance equations.

For state 1, probability flows in from state 1 with probability $1/k$ and from state 2 with probability 1, giving

$$\pi_1 = \frac{1}{k}\pi_1 + \pi_2.$$

For states $j = 2, \dots, k-1$, probability flows in from state 1 with probability $1/k$ and from state $j+1$ with probability 1, so

$$\pi_j = \frac{1}{k}\pi_1 + \pi_{j+1}.$$

For state k , probability flows in only from state 1, yielding

$$\pi_k = \frac{1}{k}\pi_1.$$

Solving these equations recursively, we obtain

$$\begin{aligned} \pi_k &= \frac{1}{k}\pi_1, \\ \pi_{k-1} &= \frac{1}{k}\pi_1 + \pi_k = \frac{2}{k}\pi_1, \\ \pi_{k-2} &= \frac{1}{k}\pi_1 + \pi_{k-1} = \frac{3}{k}\pi_1, \end{aligned}$$

and in general,

$$\pi_j = \frac{k-j+1}{k}\pi_1, \quad j = 2, \dots, k.$$

Using the normalization condition,

$$1 = \sum_{j=1}^k \pi_j = \pi_1 + \sum_{j=2}^k \frac{k-j+1}{k}\pi_1 = \pi_1 + \frac{\pi_1}{k} \sum_{m=1}^{k-1} m.$$

Since $\sum_{m=1}^{k-1} m = \frac{(k-1)k}{2}$, we have

$$1 = \pi_1 \left(1 + \frac{k-1}{2} \right) = \pi_1 \frac{k+1}{2},$$

and hence

$$\pi_1 = \frac{2}{k+1}.$$

Therefore, the stationary (and limiting) distribution is

$$\boxed{\pi_j = \frac{2}{k(k+1)}(k-j+1), \quad j = 1, \dots, k.}$$

Since the chain is ergodic, this stationary distribution is also the limiting distribution.

Exercise 3.29

Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 0.6 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \\ 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 \end{pmatrix}.$$

Identify the communication classes. Classify the states as recurrent or transient, and determine the period of each state.

Recurrent and Transient States

State j is said to be *recurrent* if the Markov chain started in j eventually revisits j . That is, $f_j = 1$.

State j is said to be *transient* if there is positive probability that the Markov chain started in j never returns to j . That is, $f_j < 1$.

Recurrence, Transience

(i) State j is recurrent if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n = \infty.$$

(ii) State j is transient if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n < \infty.$$

Recurrence and Transience are Class Properties

The states of a communication class are either all recurrent or all transient.

Period

For a Markov chain with transition matrix P , the *period* of state i , denoted $d(i)$, is the greatest common divisor of the set of possible return times to i . That is,

$$d(i) = \gcd\{n > 0 : P_{ii}^n > 0\}.$$

If $d(i) = 1$, state i is said to be *aperiodic*. If the set of return times is empty, set $d(i) = +\infty$.

Periodicity is a Class Property

The states of a communication class all have the same period.

Closed Communication Class

A communication class is *closed* if it consists of all recurrent states. A finite communication class is *closed* only if it consists of all recurrent states.

Solution

Identifying Communication Classes

From the transition graph, the communication classes can be identified as follows.

State a can transition to a , b , and e , but no other state has a path back to a . Hence, a does not communicate with any other state, and forms the singleton communication class $\{a\}$.

State e satisfies $P_{ee} = 1$, so once the chain enters e it remains there forever. Thus, e only communicates with itself and forms the singleton communication class $\{e\}$.

States b , f , and g all communicate with one another. In particular, $b \rightarrow f, g$, $f \rightarrow b, g$, and $g \rightarrow b, f$, and there are no transitions from this set to states outside the set that return back. Therefore, $\{b, f, g\}$ is a communication class.

States c and d communicate with each other since $c \rightarrow d$ and $d \rightarrow c$, and each can also return to itself with positive probability. Hence, $\{c, d\}$ forms a communication class.

Classification of States

We classify states as recurrent or transient by using the fact that recurrence and transience are *class properties*.

From the previous part, the communication classes are

$$\{a\}, \quad \{e\}, \quad \{b, f, g\}, \quad \{c, d\}.$$

By Theorem, all states in a communication class are either all recurrent or all transient. Thus, it suffices to classify each communication class.

A communication class is said to be *closed* if there are no transitions from states in the class to states outside the class. **In a finite Markov chain, every closed communication class is recurrent, and every non-closed communication class is transient.**

The class $\{e\}$ is closed, since $P_{ee} = 1$ and no transitions leave state e . Hence, state e is recurrent.

The class $\{b, f, g\}$ is also closed. Transitions among these states remain within the set, and there are no transitions from b , f , or g to any state outside the class. Therefore, states b , f , and g are recurrent.

The class $\{c, d\}$ is not closed, since state d can transition to state b , which lies outside the class, and there is no path from $\{b, f, g\}$ back to $\{c, d\}$. Thus, states c and d are transient.

Finally, the class $\{a\}$ is not closed, since state a can transition to states b and e , and no other state has a transition back to a . Hence, state a is transient.

In summary, the recurrent states are

$$\{b, f, g, e\},$$

and the transient states are

$$\{a, c, d\}.$$

Identifying the Period

We now determine the period of each state.

Recall that the period of a state i is defined by

$$d(i) = \gcd\{n > 0 : P_{ii}^n > 0\}.$$

By Lemma, periodicity is a class property, so it suffices to determine the period for each communication class.

The communication classes are

$$\{a\}, \quad \{e\}, \quad \{b, f, g\}, \quad \{c, d\}.$$

Class $\{a\}$: Since $P_{aa} = 0.6 > 0$, a return to state a is possible in one step. Hence,

$$d(a) = 1,$$

and a is aperiodic.

Class $\{e\}$: State e is absorbing, with $P_{ee} = 1$. Thus, a return to e is possible in one step, and

$$d(e) = 1.$$

Therefore, e is aperiodic.

Class $\{b, f, g\}$: From state b , a return is possible in two steps (e.g. $b \rightarrow f \rightarrow b$ or $b \rightarrow g \rightarrow b$) and in three steps (e.g. $b \rightarrow f \rightarrow g \rightarrow b$). Hence,

$$\gcd\{2, 3\} = 1,$$

and the period of this class is 1. Thus, states b , f , and g are aperiodic.

Class $\{c, d\}$: State c can return to itself in one step since $P_{cc} = 0.3 > 0$, and also in two steps via $c \rightarrow d \rightarrow c$. Therefore,

$$\gcd\{1, 2\} = 1,$$

and the period of this class is 1. Hence, states c and d are aperiodic.

Exercise 3.52

The board for a modified Snakes and Ladder game is shown in Figure 3.18. The game is played with a tetrahedron (four-faced) die.

- Find the expected length of the game.
- Assume that the player is on square 6. Find the probability that they will find themselves on square 3 before finishing the game.

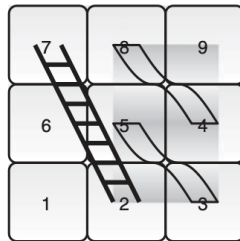


Figure 1: Figure 3.18 — the modified Snakes and Ladders board.

Solution

Part A:

Expected length of the game. Let the state space be $\{1, 2, \dots, 9\}$, with state 9 absorbing (game over), and let the transition matrix be

$$P = \begin{pmatrix} 0 & 0 & \frac{2}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $i = 1, \dots, 9$, define

$$m_i := \mathbb{E}[\text{number of moves to reach state 9} \mid X_0 = i].$$

Clearly $m_9 = 0$. By first-step analysis, for each $i \neq 9$,

$$m_i = 1 + \sum_{j=1}^9 P_{ij} m_j.$$

Using the rows of the matrix P , this gives the system

$$\begin{aligned} m_1 &= 1 + \frac{2}{4}m_3 + \frac{1}{4}m_4 + \frac{1}{4}m_7, \\ m_2 &= 1 + \frac{2}{4}m_3 + \frac{1}{4}m_4 + \frac{1}{4}m_6, \\ m_3 &= 1 + \frac{1}{4}m_3 + \frac{1}{4}m_4 + \frac{1}{4}m_6 + \frac{1}{4}m_7, \\ m_4 &= 1 + \frac{1}{4}m_3 + \frac{1}{4}m_4 + \frac{1}{4}m_6 + \frac{1}{4}m_7, \\ m_5 &= 1 + \frac{1}{4}m_4 + \frac{1}{4}m_6 + \frac{1}{4}m_7 + \frac{1}{4}m_9, \\ m_6 &= 1 + \frac{1}{4}m_4 + \frac{1}{4}m_7 + \frac{1}{2}m_9, \\ m_7 &= 1 + \frac{1}{4}m_4 + \frac{3}{4}m_9, \\ m_8 &= 1 + m_9, \\ m_9 &= 0. \end{aligned}$$

Solving yields

$$m_1 = \frac{110}{23}, \quad m_2 = \frac{113}{23}, \quad m_3 = \frac{100}{23}, \quad m_4 = \frac{100}{23}, \quad m_5 = \frac{75}{23}, \quad m_6 = \frac{60}{23}, \quad m_7 = \frac{48}{23}, \quad m_8 = 1, \quad m_9 = 0.$$

Systems of Equations Calculator: [Solution Link](#)

Therefore, starting from square 1, the expected length of the game is

$$\mathbb{E}[\text{length of the game} \mid X_0 = 1] = m_1 = \frac{110}{23} \approx 4.78 \text{ moves.}$$

Part B:

Let

$$h_i := \mathbb{P}_i(\text{hit state 3 before hitting 9}),$$

i.e., the probability that starting from square i the chain visits square 3 before finishing the game (square 9).

Then the boundary conditions are

$$h_3 = 1, \quad h_9 = 0.$$

For $i \neq 3, 9$, by first-step analysis,

$$h_i = \sum_{j=1}^9 P_{ij} h_j.$$

Using the transition matrix, we only need the equations for the states accessible from 6 before absorption:

$$\begin{aligned} h_7 &= \frac{1}{4}h_4 + \frac{3}{4}h_9, \\ h_6 &= \frac{1}{4}h_4 + \frac{1}{4}h_7 + \frac{1}{2}h_9, \\ h_4 &= \frac{1}{4}h_3 + \frac{1}{4}h_4 + \frac{1}{4}h_6 + \frac{1}{4}h_7. \end{aligned}$$

Substitute $h_3 = 1$ and $h_9 = 0$ to get

$$h_7 = \frac{1}{4}h_4, \quad h_6 = \frac{1}{4}h_4 + \frac{1}{4}h_7 = \frac{1}{4}h_4 + \frac{1}{16}h_4 = \frac{5}{16}h_4.$$

Plug these into the equation for h_4 :

$$h_4 = \frac{1}{4} + \frac{1}{4}h_4 + \frac{1}{4}\left(\frac{5}{16}h_4\right) + \frac{1}{4}\left(\frac{1}{4}h_4\right) = \frac{1}{4} + \left(\frac{1}{4} + \frac{5}{64} + \frac{1}{16}\right)h_4 = \frac{1}{4} + \frac{25}{64}h_4.$$

Hence

$$\left(1 - \frac{25}{64}\right)h_4 = \frac{1}{4} \implies \frac{39}{64}h_4 = \frac{1}{4} \implies h_4 = \frac{16}{39}.$$

Therefore

$$h_6 = \frac{5}{16}h_4 = \frac{5}{16} \cdot \frac{16}{39} = \frac{5}{39}.$$

$$\boxed{\mathbb{P}_6(\text{hit 3 before 9}) = h_6 = \frac{5}{39}.$$

Exercise 3.54

A mouse is placed in the maze in Figure 3.19 starting in box A . A piece of cheese is put in box I . From each room the mouse moves to an adjacent room through an open door, choosing from the available doors with equal probability.

- (a) How many rooms, on average, will the mouse visit before it finds the cheese?
- (b) How many times, on average, will the mouse visit room A before it finds the cheese?

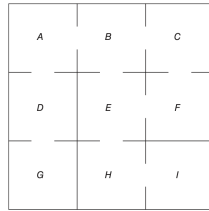


Figure 2: Figure 3.19 — the nine-room maze, with the mouse starting in A and cheese in I .

Solution

$$P = \begin{matrix} & \begin{matrix} A & B & C & D & E & F & G & H & I \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \end{matrix} & \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

Part A:

Let X_n denote the room occupied after the n th move, with state space

$$\{A, B, C, D, E, F, G, H, I\},$$

and let I be the (absorbing) cheese room. For each room x , define

$$m_x := \mathbb{E}_x[T_I], \quad T_I := \inf\{n \geq 0 : X_n = I\},$$

so m_x is the expected number of moves (equivalently, the expected number of room-visits *before* entering I) until the mouse finds the cheese when it starts in room x . Clearly $m_I = 0$.

From Figure 3.19, the open-door adjacencies are

$$A \leftrightarrow B, A \leftrightarrow D, B \leftrightarrow C, B \leftrightarrow E, C \leftrightarrow F, D \leftrightarrow G, E \leftrightarrow F, E \leftrightarrow H, H \leftrightarrow I,$$

and from each room the mouse chooses uniformly among the available doors. Thus the degrees are

$$\deg(A) = 2, \deg(B) = 3, \deg(C) = 2, \deg(D) = 2, \deg(E) = 3, \deg(F) = 2, \deg(G) = 1, \deg(H) = 2.$$

By first-step analysis, for each $x \neq I$,

$$m_x = 1 + \sum_y P_{xy} m_y,$$

which here yields the linear system

$$\begin{aligned} m_A &= 1 + \frac{1}{2}m_B + \frac{1}{2}m_D, \\ m_B &= 1 + \frac{1}{3}(m_A + m_C + m_E), \\ m_C &= 1 + \frac{1}{2}(m_B + m_F), \\ m_D &= 1 + \frac{1}{2}(m_A + m_G), \\ m_E &= 1 + \frac{1}{3}(m_B + m_F + m_H), \\ m_F &= 1 + \frac{1}{2}(m_C + m_E), \\ m_G &= 1 + m_D, \\ m_H &= 1 + \frac{1}{2}(m_E + m_I), \\ m_I &= 0. \end{aligned}$$

Solving gives

$$m_A = \frac{89}{2}, \quad m_B = \frac{79}{2}, \quad m_C = 39, \quad m_D = \frac{95}{2}, \quad m_E = 32, \quad m_F = \frac{73}{2}, \quad m_G = \frac{97}{2}, \quad m_H = 17, \quad m_I = 0.$$

Therefore, starting from A , the expected number of rooms visited before the mouse finds the cheese is

$$m_A = \frac{89}{2} = 44.5.$$

Part B:

As in part (a), treat room I as absorbing (once the mouse reaches I , the game ends). For each room $x \in \{A, B, C, D, E, F, G, H, I\}$ define

$$v_x := \mathbb{E}_x[\#\{n < T_I : X_n = A\}], \quad T_I := \inf\{n \geq 0 : X_n = I\}.$$

Thus v_x is the expected number of *visits to A* before the cheese is found, starting from x . In particular, the desired quantity is v_A . (With this definition, the starting position counts as a visit, so starting in A already contributes 1.) Clearly,

$$v_I = 0.$$

By first-step analysis, for each $x \neq I$,

$$v_x = \mathbf{1}\{x = A\} + \sum_y P_{xy} v_y,$$

where $\mathbf{1}\{x = A\} = 1$ if $x = A$ and 0 otherwise.

Using the open-door transitions from Figure 3.19 (uniform among available doors), we obtain the linear system

$$\begin{aligned} v_A &= 1 + \frac{1}{2}v_B + \frac{1}{2}v_D, \\ v_B &= \frac{1}{3}(v_A + v_C + v_E), \\ v_C &= \frac{1}{2}(v_B + v_F), \\ v_D &= \frac{1}{2}(v_A + v_G), \\ v_E &= \frac{1}{3}(v_B + v_F + v_H), \\ v_F &= \frac{1}{2}(v_C + v_E), \\ v_G &= v_D, \\ v_H &= \frac{1}{2}(v_E + v_I) = \frac{1}{2}v_E, \\ v_I &= 0. \end{aligned}$$

Solving yields

$$v_A = \frac{15}{2}, \quad v_B = \frac{11}{2}, \quad v_C = 5, \quad v_D = \frac{15}{2}, \quad v_E = 4, \quad v_F = \frac{9}{2}, \quad v_G = \frac{15}{2}, \quad v_H = 2, \quad v_I = 0.$$

Therefore, starting from A , the expected number of visits to room A before the mouse finds the cheese is

$$v_A = \frac{15}{2} = 7.5.$$

Exercise 3.66

R: Simulate the expected hitting time for the random walk on the hexagon in Exercise 3.19.

Solution

Input:

Listing 1: Monte Carlo Simulation

```

1 # Random walk on the graph in Figure 3.15
2 # Estimate E_a[T_d] = expected hitting time to reach d starting from a
3
4 set.seed(123)
5
6 # --- Graph (neighbors) read from the figure ---
7 # Edges: a-b, b-c, c-d, d-e, e-f, f-a plus chords a-c, a-e, c-e
8 neighbors <- list(
9   a = c("b", "f", "c", "e"),
10  b = c("a", "c"),
11  c = c("b", "d", "a", "e"),
12  d = c("c", "e"),
13  e = c("d", "f", "a", "c"),
14  f = c("a", "e")
15 )
16
17 rw_hitting_time <- function(start = "a", target = "d", neighbors) {
18   state <- start
19   t <- 0L
20   while (state != target) {
21     state <- sample(neighbors[[state]], size = 1)
22     t <- t + 1L
23   }
24   t
25 }
26
27 simulate_expected_hitting_time <- function(n_sims = 10000, start = "a", target = "d",
28   neighbors) {
29   times <- replicate(n_sims, rw_hitting_time(start, target, neighbors))
30   list(
31     estimate = mean(times),
32     sd = sd(times),
33     se = sd(times) / sqrt(n_sims),
34     ci95 = mean(times) + c(-1, 1) * 1.96 * sd(times) / sqrt(n_sims)
35   )
36 }
37
38 # --- Run the simulation ---
39 res <- simulate_expected_hitting_time(n_sims = 50000, start = "a", target = "d",
40   neighbors = neighbors)
41 res

```

Output: 95% Confidence Interval of (9.909, 10.059) where the units are steps. This matches up nicely with the exact value of 10 steps, which can be seen by solving the 3x3 linear system hinted in 3.19.