

STAT 545 HW 2

Andres Efren Rocha Jayasinha

Updated: September 20, 2026

Table of Contents

Assignment 2: Limiting and Stationary Distributions	2
Exercise 3.7	2
Exercise 3.8	3
Exercise 3.10	5
Exercise 3.14	6
Exercise 3.63	7

Assignment 2: Limiting and Stationary Distributions

Exercise 3.7

A Markov chain has n states. If the chain is at state k , a coin is flipped, whose heads probability is p . If the coin lands heads, the chain stays at k . If the coin lands tails, the chain moves to a different state uniformly at random. Exhibit the transition matrix and find the stationary distribution.

Stationary Distribution. Let $X_0, X_1, \dots$ be a Markov chain with transition matrix P . A *stationary distribution* is a probability distribution π that satisfies

$$\pi = \pi P.$$

That is,

$$\pi_j = \sum_i \pi_i P_{ij}, \quad \text{for all } j.$$

Solution

A stationary distribution π satisfies

$$\pi = \pi P,$$

where π is a row vector of dimension $(1 \times n)$ and P is an $(n \times n)$ transition matrix.

Writing this product explicitly,

$$(\pi_1, \dots, \pi_n) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} = (\pi_1, \dots, \pi_n).$$

In this chain,

$$p_{ij} = \begin{cases} p, & i = j, \\ \frac{1-p}{n-1}, & i \neq j. \end{cases}$$

For each fixed j , the j th coordinate of πP is

$$\pi_j = \sum_{i=1}^n \pi_i p_{ij} = \pi_j p + \sum_{i \neq j} \pi_i \frac{1-p}{n-1}.$$

Rearranging,

$$\pi_j(1-p) = \frac{1-p}{n-1} \sum_{i \neq j} \pi_i.$$

Since $1-p > 0$, dividing both sides by $1-p$ gives

$$\pi_j = \frac{1}{n-1} \sum_{i \neq j} \pi_i.$$

Using $\sum_{i=1}^n \pi_i = 1$, we have $\sum_{i \neq j} \pi_i = 1 - \pi_j$, hence

$$\pi_j = \frac{1 - \pi_j}{n-1}.$$

Solving,

$$(n-1)\pi_j = 1 - \pi_j \quad \Rightarrow \quad n\pi_j = 1 \quad \Rightarrow \quad \pi_j = \frac{1}{n}.$$

Therefore, the stationary distribution is

$$\pi = \left(\frac{1}{n}, \dots, \frac{1}{n} \right).$$

Exercise 3.8

Let

$$P_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}.$$

Consider a Markov chain on four states whose transition matrix is given by the block matrix

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

- (a) Does the Markov chain have a unique stationary distribution? If so, find it.
- (b) Does $\lim_{n \rightarrow \infty} P^n$ exist? If so, find it.
- (c) Does the Markov chain have a limiting distribution? If so, find it.

Limiting Distribution.

Let $X_0, X_1, \dots$ be a Markov chain with transition matrix P . A *limiting distribution* for the Markov chain is a probability distribution λ with the property that, for all i and j ,

$$\lim_{n \rightarrow \infty} P_{ij}^n = \lambda_j.$$

Limiting Distributions are Stationary Distributions.

Assume that π is the limiting distribution of a Markov chain with transition matrix P . Then π is a stationary distribution.

Positive Matrices and Vectors.

A matrix M is said to be *positive* if all entries of M are positive. In this case, we write $M > 0$. Similarly, we write $x > 0$ for a vector x whose entries are all positive.

Note. The notation $M > 0$ (or $x > 0$) means *entrywise positivity*, that is, every entry is strictly greater than zero. This is stronger than nonnegativity and does not refer to positive definiteness.

Regular Transition Matrix.

A transition matrix P is said to be *regular* if some power of P is positive. That is,

$$P^n > 0 \quad \text{for some } n \geq 1.$$

Limit Theorem for Regular Markov Chains.

Let $X_0, X_1, \dots$ be a Markov chain whose transition matrix P is regular. Then the chain has a limiting distribution, which is the unique, positive stationary distribution of the chain. That is, there exists a unique probability vector $\pi > 0$ such that

$$\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j, \quad \text{for all } i, j,$$

where π satisfies

$$\sum_i \pi_i P_{ij} = \pi_j.$$

Equivalently, there exists a positive stochastic matrix Π such that

$$\lim_{n \rightarrow \infty} P^n = \Pi,$$

where Π has equal rows with common row π , and π is the unique probability vector satisfying

$$\pi P = \pi.$$

Solution

Part A:

A stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4) = ((\pi_1, \pi_2), (\pi_3, \pi_4))$$

must satisfy $\pi = \pi P$. Since

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

this is equivalent to requiring that:

- (π_1, π_2) is a stationary distribution for P_1 ,
- (π_3, π_4) is a stationary distribution for P_2 ,
- $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

We first find the stationary distribution of P_1 . Let (a, b) satisfy $(a, b) = (a, b)P_1$ with $a + b = 1$. Then

$$a = \frac{1}{4}a + \frac{1}{2}b \implies \frac{3}{4}a = \frac{1}{2}b \implies 3a = 2b.$$

Together with $a + b = 1$, this yields $a = \frac{2}{5}$ and $b = \frac{3}{5}$. Hence,

$$\pi^{(1)} = \left(\frac{2}{5}, \frac{3}{5} \right).$$

Next, we find the stationary distribution of P_2 . By symmetry of the transition matrix P_2 , the stationary distribution is

$$\pi^{(2)} = \left(\frac{1}{2}, \frac{1}{2} \right).$$

Therefore, any stationary distribution of P is a combination of these two stationary distributions. Let $\alpha \in [0, 1]$ denote the total stationary mass assigned to states $\{1, 2\}$. Then all stationary distributions are given by

$$\pi(\alpha) = (\alpha \pi^{(1)}, (1 - \alpha) \pi^{(2)}) = \left(\alpha \frac{2}{5}, \alpha \frac{3}{5}, (1 - \alpha) \frac{1}{2}, (1 - \alpha) \frac{1}{2} \right), \quad 0 \leq \alpha \leq 1.$$

Consequently, the Markov chain does *not* have a unique stationary distribution.

Part B:

Since

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

it follows by block-matrix multiplication that for every integer $n \geq 1$,

$$P^n = \begin{pmatrix} P_1^n & 0 \\ 0 & P_2^n \end{pmatrix}.$$

Hence, $\lim_{n \rightarrow \infty} P^n$ exists if and only if both limits $\lim_{n \rightarrow \infty} P_1^n$ and $\lim_{n \rightarrow \infty} P_2^n$ exist, in which case

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \lim_{n \rightarrow \infty} P_1^n & 0 \\ 0 & \lim_{n \rightarrow \infty} P_2^n \end{pmatrix}.$$

All entries of P_1 are strictly positive, so P_1 is a regular transition matrix. By the Limit Theorem for Regular Markov Chains, P_1 has a unique positive stationary distribution, specifically $\pi^{(1)} = (\frac{3}{5}, \frac{2}{5})$ as we showed in Part A,

$$\implies \lim_{n \rightarrow \infty} P_1^n = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}.$$

Similarly, all entries of P_2 are strictly positive, so P_2 is also regular. By the Limit Theorem for Regular Markov Chains, P_2 has a unique positive stationary distribution, specifically $\pi^{(2)} = (\frac{1}{2}, \frac{1}{2})$ as we showed in Part A,

$$\implies \lim_{n \rightarrow \infty} P_2^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Combining the two block limits, we conclude that

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Thus, $\lim_{n \rightarrow \infty} P^n$ exists and is given by the matrix above.

Part C:

Recall that a limiting distribution λ must satisfy

$$\lim_{n \rightarrow \infty} P_{ij}^n = \lambda_j \quad \text{for all } i, j,$$

which is equivalent to requiring that every row of P^n converges to the same probability vector.

From part (b), we computed

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The first two rows converge to $(\frac{2}{5}, \frac{3}{5}, 0, 0)$, while the last two rows converge to $(0, 0, \frac{1}{2}, \frac{1}{2})$. Since these limiting rows are not equal, there is no single probability vector λ such that

$$\lim_{n \rightarrow \infty} P_{ij}^n = \lambda_j \quad \text{for all } i.$$

Therefore, the Markov chain does not have a limiting distribution.

Exercise 3.10

A Markov chain has transition matrix P and limiting distribution π . Further assume that π is the initial distribution of the chain; that is, the chain is in stationarity. Find the following:

- (a) $\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_{n-1} = i)$
- (b) $\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_0 = i)$
- (c) $\lim_{n \rightarrow \infty} \mathbb{P}(X_{n+1} = k, X_n = j \mid X_0 = i)$
- (d) $\lim_{n \rightarrow \infty} \mathbb{P}(X_0 = j \mid X_n = i)$

Solution

Part A:

For each $n \geq 1$, by time-homogeneity of the Markov chain we have

$$\mathbb{P}(X_n = j \mid X_{n-1} = i) = \mathbb{P}(X_1 = j \mid X_0 = i) = P_{ij}.$$

Since this quantity does not depend on n , it follows that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_{n-1} = i) = P_{ij}.$$

Part B:

For each $n \geq 0$,

$$\mathbb{P}(X_n = j \mid X_0 = i) = (P^n)_{ij}.$$

Since the chain has limiting distribution π , by definition

$$\lim_{n \rightarrow \infty} (P^n)_{ij} = \pi_j \quad \text{for all } i, j.$$

Therefore,

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_0 = i) = \lim_{n \rightarrow \infty} (P^n)_{ij} = \pi_j.$$

Part C:

For $n \geq 0$,

$$\mathbb{P}(X_{n+1} = k, X_n = j \mid X_0 = i) = \mathbb{P}(X_n = j \mid X_0 = i) \mathbb{P}(X_{n+1} = k \mid X_n = j, X_0 = i).$$

By the Markov property,

$$\mathbb{P}(X_{n+1} = k \mid X_n = j, X_0 = i) = \mathbb{P}(X_{n+1} = k \mid X_n = j) = P_{jk},$$

and also

$$\mathbb{P}(X_n = j \mid X_0 = i) = (P^n)_{ij}.$$

Hence

$$\mathbb{P}(X_{n+1} = k, X_n = j \mid X_0 = i) = (P^n)_{ij} P_{jk}.$$

Since the chain has limiting distribution π , we have

$$\lim_{n \rightarrow \infty} (P^n)_{ij} = \pi_j,$$

and therefore

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_{n+1} = k, X_n = j \mid X_0 = i) = \pi_j P_{jk}.$$

Part D:

By Bayes' rule,

$$\mathbb{P}(X_0 = j \mid X_n = i) = \frac{\mathbb{P}(X_n = i \mid X_0 = j) \mathbb{P}(X_0 = j)}{\mathbb{P}(X_n = i)}.$$

Since the chain is started in stationarity, $\mathbb{P}(X_0 = j) = \pi_j$ and $\mathbb{P}(X_n = i) = \pi_i$ for all n . Also,

$$\mathbb{P}(X_n = i \mid X_0 = j) = (P^n)_{ji}.$$

Therefore,

$$\mathbb{P}(X_0 = j \mid X_n = i) = \frac{(P^n)_{ji} \pi_j}{\pi_i}.$$

Because the chain has limiting distribution π , we have

$$\lim_{n \rightarrow \infty} (P^n)_{ji} = \pi_i,$$

and hence

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_0 = j \mid X_n = i) = \lim_{n \rightarrow \infty} \frac{(P^n)_{ji} \pi_j}{\pi_i} = \frac{\pi_i \pi_j}{\pi_i} = \pi_j,$$

Exercise 3.14

The California Air Resources Board warns the public when smog levels are above certain thresholds. Days on which the board issues warnings are called *episode days*. Lin (1981) models the daily sequence of episode and nonepisode days as a Markov chain with transition matrix

$$P = \begin{pmatrix} 0.77 & 0.23 \\ 0.24 & 0.76 \end{pmatrix},$$

where the rows and columns correspond to the states *Nonepisode* and *Episode*, respectively.

(a) What is the long-term probability that a given day will be an episode day?

(b) Over a year's time, about how many days are expected to be episode days?

Solution

Part A:

Let $\pi = (\pi_N, \pi_E)$ denote the stationary distribution of the Markov chain, where π_N is the long-term probability of a nonepisode day and π_E is the long-term probability of an episode day.

The stationary distribution satisfies

$$\pi = \pi P, \quad \pi_N + \pi_E = 1.$$

Using the transition matrix

$$P = \begin{pmatrix} 0.77 & 0.23 \\ 0.24 & 0.76 \end{pmatrix},$$

we obtain the equations

$$\pi_E = 0.23 \pi_N + 0.76 \pi_E.$$

Rewriting,

$$\pi_E - 0.76 \pi_E = 0.23 \pi_N \implies 0.24 \pi_E = 0.23 \pi_N.$$

Using $\pi_N = 1 - \pi_E$, we have

$$0.24 \pi_E = 0.23(1 - \pi_E),$$

which implies

$$0.47 \pi_E = 0.23.$$

Thus,

$$\pi_E = \frac{0.23}{0.47} \approx 0.489.$$

Therefore, the long-term probability that a given day will be an episode day is approximately

$$\boxed{0.489}$$

Part B:

From part (a), the long-term probability that a day is an episode day is

$$\pi_E = \frac{0.23}{0.47} \approx 0.489.$$

Over a period of 365 days, the expected number of episode days is therefore

$$\mathbb{E}[\# \text{ episode days in a year}] \approx 365 \pi_E = 365 \left(\frac{0.23}{0.47} \right) \approx 365(0.489) \approx 178.617$$

So, over a year's time we expect about

$$\boxed{179 \text{ episode days}}$$

Exercise 3.63

Hourly wind speeds in a northwestern region of Turkey are modeled by a Markov chain in Sahin and Sen (2001). Seven wind speed levels are the states of the chain. The transition matrix is

$$P = \begin{pmatrix} 0.756 & 0.113 & 0.129 & 0.002 & 0 & 0 & 0 \\ 0.174 & 0.821 & 0.004 & 0.001 & 0 & 0 & 0 \\ 0.141 & 0.001 & 0.776 & 0.082 & 0 & 0 & 0 \\ 0.003 & 0 & 0.192 & 0.753 & 0.052 & 0 & 0 \\ 0 & 0 & 0.002 & 0.227 & 0.735 & 0.036 & 0 \\ 0 & 0 & 0 & 0.007 & 0.367 & 0.604 & 0.022 \\ 0 & 0 & 0 & 0 & 0.053 & 0.158 & 0.789 \end{pmatrix}$$

(a) Find the limiting distribution by

- (i) taking high matrix powers, and
- (ii) using the `stationary` command in the `utilities.R` file.

How often does the highest wind speed occur? How often does the lowest speed occur?

(b) Simulate the chain for 100,000 steps and estimate the proportion of times that the chain visits each state.

Solution

Part A:

Input:

Listing 1: Hourly Wind Speeds of a Region In Turkey

```
1 ## Part A
2
3 # Part i
4
5 # Transition matrix
6 P <- matrix(c(
7   0.756, 0.113, 0.129, 0.002, 0.000, 0.000, 0.000,
8   0.174, 0.821, 0.004, 0.001, 0.000, 0.000, 0.000,
9   0.141, 0.001, 0.776, 0.082, 0.000, 0.000, 0.000,
10  0.003, 0.000, 0.192, 0.753, 0.052, 0.000, 0.000,
11  0.000, 0.000, 0.002, 0.227, 0.735, 0.036, 0.000,
12  0.000, 0.000, 0.000, 0.007, 0.367, 0.604, 0.022,
13  0.000, 0.000, 0.000, 0.000, 0.053, 0.158, 0.789
14 ), nrow = 7, byrow = TRUE)
```

```

15
16 # Compute a high power of P
17 Pn <- P
18 for (i in 1:1000) {
19   Pn <- Pn %*% P
20 }
21
22 # Display the result
23 round(Pn, 6)
24
25 # Extract the limiting distribution (any row)
26 pi_hat <- Pn[1, ]
27 pi_hat
28
29 # Frequencies of lowest and highest wind speed states
30 pi_hat[1] # lowest wind speed
31 pi_hat[7] # highest wind speed
32
33 # Part ii
34
35 source("~/Downloads/utilities.R")
36
37 station_dist <- stationary(P)

```

Output:

The Limiting Distribution is (three decimal places): (0.324 0.206 0.303 0.131 0.029 0.002 0.000)
The highest wind speed occurs 0% of the time.
The lowest wind speed occurs 32.4% of the time.

Part B:

Input:

Listing 2: Hourly Wind Speeds of a Region In Turkey

```

1 ## Part B
2
3 P <- matrix(c(
4   0.756, 0.113, 0.129, 0.002, 0.000, 0.000, 0.000,
5   0.174, 0.821, 0.004, 0.001, 0.000, 0.000, 0.000,
6   0.141, 0.001, 0.776, 0.082, 0.000, 0.000, 0.000,
7   0.003, 0.000, 0.192, 0.753, 0.052, 0.000, 0.000,
8   0.000, 0.000, 0.002, 0.227, 0.735, 0.036, 0.000,
9   0.000, 0.000, 0.000, 0.007, 0.367, 0.604, 0.022,
10  0.000, 0.000, 0.000, 0.000, 0.053, 0.158, 0.789
11 ), nrow = 7, byrow = TRUE)
12
13 # Compute a high power of P
14 Pn <- P
15 for (i in 1:100000) {
16   Pn <- Pn %*% P
17 }
18
19 P[1, ]

```

Output:

Each row of the matrix P_n converges to (three decimal places): (0.324 0.206 0.303 0.131 0.029 0.002 0.000)