

STAT 545 HW 1

Andres Efren Rocha Jayasinha

Updated: September 20, 2026

Table of Contents

Assignment 1: Introduction to Markov Chains	2
Exercise 2.1	2
Exercise 2.9	4
Exercise 2.10	4
Exercise 2.16	5
Exercise 2.27	6

Assignment 1: Introduction to Markov Chains

Exercise 2.1

A Markov chain has transition matrix

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix},$$

with initial distribution $\alpha = (0.2, 0.3, 0.5)$. Find the following:

- (a) $P(X_7 = 3 \mid X_6 = 2)$
- (b) $P(X_9 = 2 \mid X_1 = 2, X_5 = 1, X_7 = 3)$
- (c) $P(X_0 = 3 \mid X_1 = 1)$
- (d) $E(X_2)$

Stochastic Process. A *stochastic process* is a collection of random variables $\{X_t : t \in I\}$. The set I is called the *index set* of the process. The random variables are defined on a common *state space* S .

Markov Chain. Let S be a discrete set. A *Markov chain* is a sequence of random variables $X_0, X_1, \dots$ taking values in S with the property that

$$P(X_{n+1} = j \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i),$$

for all $x_0, \dots, x_{n-1}, i, j \in S$ and all $n \geq 0$. The set S is called the *state space* of the Markov chain.

A Markov chain is said to be *time-homogeneous* if the transition probabilities do not depend on n , that is,

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i), \quad \text{for all } i, j \in S \text{ and } n \geq 0.$$

Bayes. Let $A_1, A_2, \dots, A_k$ be events such that $P(A_i) > 0, i = 1, 2, \dots, k$. Assume further that $A_1, A_2, \dots, A_k$ form a partition of the sample space $\mathcal{C}$. Let B be any event. Then

$$P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^k P(A_i)P(B \mid A_i)}, \quad (1.4.3)$$

Law of Total Probability. Let $A_1, A_2, \dots, A_k$ be events such that $P(A_i) > 0$ for $i = 1, 2, \dots, k$, and assume $A_1, \dots, A_k$ form a partition of the sample space $\mathcal{C}$. Then for any event B ,

$$P(B) = \sum_{i=1}^k P(A_i)P(B \mid A_i).$$

Solution

(a) For a time-homogeneous Markov chain,

$$P(X_{n+1} = j \mid X_n = i) = p_{ij}.$$

Therefore,

$$P(X_7 = 3 \mid X_6 = 2) = p_{2,3}.$$

From the transition matrix, the entry in row 2 and column 3 is 0.6. Hence,

$P(X_7 = 3 \mid X_6 = 2) = 0.6.$

(b) By the Markov property, conditioning on X_7 makes the earlier states X_1 and X_5 irrelevant. Thus,

$$P(X_9 = 2 \mid X_1 = 2, X_5 = 1, X_7 = 3) = P(X_9 = 2 \mid X_7 = 3).$$

Using the law of total probability and conditioning on the possible values of X_8 ,

$$P(X_9 = 2 \mid X_7 = 3) = \sum_{j=1}^3 P(X_9 = 2 \mid X_8 = j, X_7 = 3) P(X_8 = j \mid X_7 = 3).$$

By the Markov property,

$$P(X_9 = 2 \mid X_8 = j, X_7 = 3) = P(X_9 = 2 \mid X_8 = j) = p_{j2},$$

and

$$P(X_8 = j \mid X_7 = 3) = p_{3j}.$$

Therefore,

$$P(X_9 = 2 \mid X_7 = 3) = \sum_{j=1}^3 p_{3j} p_{j2}.$$

Substituting the values from the transition matrix (or multiply second column by third row),

$$= 0.3(0.3) + 0.2(0.4) + 0.5(0.2) = 0.09 + 0.08 + 0.10 = 0.27.$$

Hence,

$$\boxed{P(X_9 = 2 \mid X_1 = 2, X_5 = 1, X_7 = 3) = 0.27.}$$

(c) We want $P(X_0 = 3 \mid X_1 = 1)$. By Bayes' rule,

$$P(X_0 = 3 \mid X_1 = 1) = \frac{P(X_1 = 1 \mid X_0 = 3) P(X_0 = 3)}{P(X_1 = 1)}.$$

From the initial distribution $\alpha = (0.2, 0.3, 0.5)$, we have $P(X_0 = 3) = 0.5$. Also, from the transition matrix,

$$P(X_1 = 1 \mid X_0 = 3) = p_{3,1} = 0.3.$$

Next,

$$P(X_1 = 1) = \sum_{i=1}^3 P(X_1 = 1 \mid X_0 = i) P(X_0 = i) = \sum_{i=1}^3 p_{i1} \alpha_i.$$

Using column 1 of P , $(p_{11}, p_{21}, p_{31}) = (0.1, 0, 0.3)$, we get

$$P(X_1 = 1) = 0.1(0.2) + 0(0.3) + 0.3(0.5) = 0.02 + 0 + 0.15 = 0.17.$$

Therefore,

$$P(X_0 = 3 \mid X_1 = 1) = \frac{0.3 \cdot 0.5}{0.17} = \frac{0.15}{0.17} = \frac{15}{17} \approx 0.88235.$$

$$\boxed{P(X_0 = 3 \mid X_1 = 1) = \frac{15}{17} \approx 0.88235.}$$

(d) To compute $E(X_2)$, first find the distribution of X_2 . Since the initial distribution is $\alpha = (0.2, 0.3, 0.5)$, we have

$$\mathbb{P}(X_2 = \cdot) = \alpha P^2.$$

Compute

$$P^2 = \begin{pmatrix} 0.19 & 0.27 & 0.54 \\ 0.18 & 0.28 & 0.54 \\ 0.18 & 0.27 & 0.55 \end{pmatrix}.$$

Therefore,

$${}_aP^2 = (0.2, 0.3, 0.5) \begin{pmatrix} 0.19 & 0.27 & 0.54 \\ 0.18 & 0.28 & 0.54 \\ 0.18 & 0.27 & 0.55 \end{pmatrix} = (0.182, 0.273, 0.545).$$

So

$$\mathbb{P}(X_2 = 1) = 0.182, \quad \mathbb{P}(X_2 = 2) = 0.273, \quad \mathbb{P}(X_2 = 3) = 0.545.$$

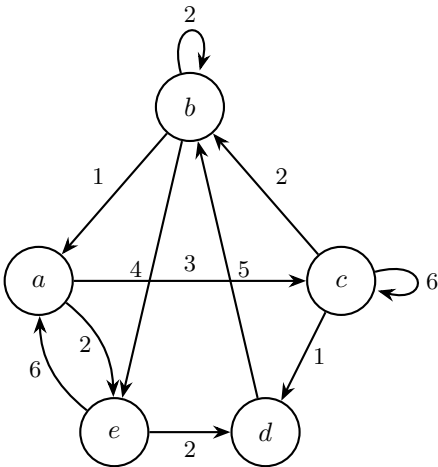
Finally,

$$E(X_2) = \sum_{k=1}^3 k \mathbb{P}(X_2 = k) = 1(0.182) + 2(0.273) + 3(0.545) = 0.182 + 0.546 + 1.635 = 2.363.$$

$E(X_2) = 2.363.$

Exercise 2.9

Give the transition matrix for the transition graph in Figure 2.11



Solution

Outgoing totals: $\deg^+(a) = 3 + 2 = 5, \quad \deg^+(b) = 1 + 2 + 4 = 7,$
 $\deg^+(c) = 2 + 6 + 1 = 9, \quad \deg^+(d) = 5, \quad \deg^+(e) = 6 + 2 = 8.$

Transition probabilities: $P_{ij} = \frac{w(i \rightarrow j)}{\sum_k w(i \rightarrow k)}.$

$$\Rightarrow \text{Transition matrix } P \text{ (order } (a, b, c, d, e)): \quad P = \begin{pmatrix} 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{1}{7} & \frac{2}{7} & 0 & 0 & \frac{4}{7} \\ 0 & \frac{2}{9} & \frac{2}{3} & \frac{1}{9} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

Exercise 2.10

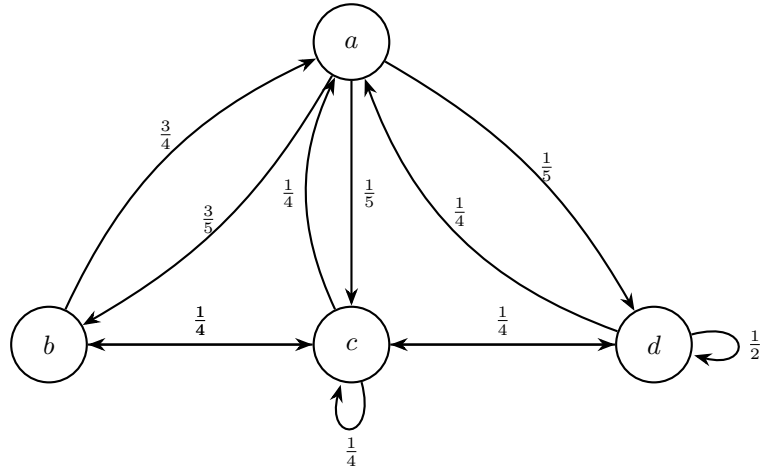
Consider a Markov chain with transition matrix

$$P = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ b & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ c & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ d & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{array}$$

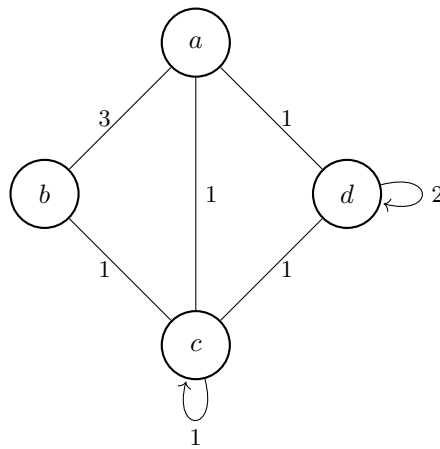
- Exhibit the directed, weighted transition graph for the chain.
- The transition graph for this chain can be given as a weighted graph without directed edges. Exhibit the graph.

Solution

(a)



(b)



Exercise 2.16

Assume that P is a stochastic matrix with equal rows. Show that

$$P^n = P, \quad \text{for all } n \geq 1.$$

(Stochastic Matrix). A stochastic matrix is a square matrix P such that:

1. $P_{ij} \geq 0$ for all i, j .
2. For each row i ,

$$\sum_j P_{ij} = 1.$$

Solution

Assume that P is a stochastic matrix whose rows are all equal. Let r^T denote this common row, so that $P_{ij} = r_j$ for all i, j . Since P is stochastic, the entries of r satisfy

$$\sum_k r_k = 1.$$

We will prove by induction for $n \geq 1$ that $P^n = P$.

Base case ($n = 1$): Clearly,

$$P^1 = P.$$

Inductive hypothesis: Assume that for some $k \geq 1$,

$$P^k = P.$$

Inductive step: Using the inductive hypothesis,

$$P^{k+1} = P^k P = PP = P^2.$$

If we now show that $P^2 = P$, then the proof is complete. For any i, j ,

$$(P^2)_{ij} = \sum_m P_{im} P_{mj} = \sum_m r_m r_j = r_j \sum_m r_m = r_j = P_{ij},$$

where we used the fact that $\sum_m r_m = 1$. Thus $P^2 = P$, and hence

$$P^{k+1} = P.$$

By mathematical induction,

$$P^n = P \quad \text{for all } n \geq 1.$$

Exercise 2.27

R: See `gamblersruin.R`. Simulate gambler's ruin for a gambler with initial stake \$2, playing a fair game.

- Estimate the probability that the gambler is ruined before he wins \$5.
- Construct the transition matrix for the associated Markov chain. Estimate the desired probability in (a) by taking high matrix powers.
- Compare your results with the exact probability.

Solution

(a)

Input:

Listing 1: Gambler's Ruin Simulation in R

```

1 # gamblersruin.R
2 # Example 1.11
3
4 # gamble(k, n, p)
5 #   k: Gambler's initial state
6 #   n: Gambler plays until either n or Ruin
7 #   p: Probability of winning $1 at each play
8 #   Function returns 1 if gambler is eventually ruined
9 #   returns 0 if gambler eventually wins n
10
11 gamble <- function(k, n, p) {
12   stake <- k
13   while (stake > 0 & stake < n) {
14     bet <- sample(c(-1, 1), 1, prob = c(1 - p, p))
15     stake <- stake + bet
16   }
17   if (stake == 0) return(1) else return(0)
18 }
19
20 k <- 2
21 n <- 40
22 p <- 1/2
23 trials <- 1000
24
25 simlist <- replicate(trials, gamble(k, n, p))
26 mean(simlist) # Estimate of probability that gambler is ruined
27
28 # For p = 0.5, exact probability is (n - k) / n

```

Output: *Estimated Probability of Ruin* = 0.952

(b)

Input:

Listing 2: Gambler's Ruin Simulation in R

```
1 k <- 2
2 n <- 40
3 p <- 1/2
4 q <- 1 - p
5
6 P <- matrix(0, n+1, n+1)
7
8 # absorbing states
9 P[1,1] <- 1 # state 0
10 P[n+1, n+1] <- 1 # state n
11
12 # interior transitions
13 for (i in 1:(n-1)) {
14   P[i+1, i] <- q # from i to i-1
15   P[i+1, i+2] <- p # from i to i+1
16 }
17
18 # high power
19 t <- 2000
20 Pt <- diag(n+1)
21 for (j in 1:t) Pt <- Pt %*% P
22
23 ruin_prob_est <- Pt[k+1, 1] # (P^t)_{k,0}
24 win_prob_est <- Pt[k+1, n+1] # (P^t)_{k,n}
25
26 ruin_prob_est
27 win_prob_est
28 ruin_prob_est + win_prob_est # should be ~1
```

Output: Estimated Probability of Ruin = 0.949

Note that the transition matrix P is,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & q & 0 & p & 0 \\ 0 & \cdots & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with $p = \frac{1}{2}$ and $q = \frac{1}{2}$

(c) From `gamblersruin.R` we have,

$$\begin{aligned} \text{Exact Probability of Ruin} &= \frac{n-k}{n} \\ \implies \text{Exact Probability of Ruin} &= \frac{40-2}{40} = \boxed{0.95} \end{aligned}$$

Given that the estimated probabilities of ruin were 0.952, 0.949, they match up nicely with the exact probability of ruin, that being 0.95.